

Fidelity approach to quantum phase transitions: finite-size scaling for the quantum Ising model in a transverse field

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 492002

(<http://iopscience.iop.org/1751-8121/41/49/492002>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.152

The article was downloaded on 03/06/2010 at 07:21

Please note that [terms and conditions apply](#).

FAST TRACK COMMUNICATION

Fidelity approach to quantum phase transitions: finite-size scaling for the quantum Ising model in a transverse field

Huan-Qiang Zhou, Jian-Hui Zhao and Bo Li

Centre for Modern Physics and Department of Physics, Chongqing University,
Chongqing 400044, People's Republic of China

Received 17 September 2008

Published 29 October 2008

Online at stacks.iop.org/JPhysA/41/492002

Abstract

We analyze the fidelity per lattice site for two different ground states of the one-dimensional quantum Ising model in a transverse field near the critical point. It is found that, in the thermodynamic limit, the fidelity per lattice site is singular, and the derivative of its logarithmic function with respect to the transverse field strength is logarithmically divergent at the critical point. The scaling behavior is confirmed numerically by performing a finite-size scaling analysis for systems of different sizes, consistent with the conformal invariance at the critical point. This allows us to extract the correlation length critical exponent, which turns out to be universal in the sense that the correlation length critical exponent does not depend on either the anisotropic parameter or the transverse field strength.

PACS numbers: 03.67.-a, 05.70.Fh, 64.60.Ak

(Some figures in this article are in colour only in the electronic version)

Introduction

An emerging picture arises due to the latest advances in quantum information science, which allows us to study quantum phase transitions (QPTs) [1] from the ground-state wavefunctions of many-body systems. One of the well-studied aspects is to unveil the possible role of entanglement in characterizing QPTs [2–7] (for a review, see [8]). Remarkably, for quantum spin chains, the von Neumann entropy, as a bipartite entanglement measure, exhibits qualitatively different behaviors at and off criticality [4].

On the other hand, the fidelity, another basic notion of quantum information science, has attracted a lot of attention [9–11] quite recently. In [10], it has been shown that it may be used to characterize QPTs, which occur in quantum spin chains, regardless of what type

of internal order is present in quantum many-body states (either the conventional symmetry-broken orders or exotic QPTs in matrix product systems [12]). The argument is solely based on the basic postulate of quantum mechanics on quantum measurements. Indeed, the basic postulate of quantum mechanics on quantum measurements implies that two non-orthogonal quantum states are not reliably distinguishable [13]. Therefore, any two ground states must be orthogonal due to the occurrence of orders, regardless of what type of QPTs. Conversely, the fact that two ground states are orthogonal implies that they are reliably distinguishable. Therefore, an order parameter, which may be constructed systematically in principle [14], exists for any systems undergoing QPTs. It is the quantitative or qualitative difference unveiled in order parameters that justifies the introduction of the notions of irrelevant and relevant information. To quantify irrelevant and relevant information, the scaling parameter extracted from the fidelity, i.e. the fidelity per lattice site, was introduced to characterize QPTs. This establishes an intriguing connection between quantum information theory, QPTs, renormalization group (RG) flows and condensed matter physics.

The fact that any two different ground states are orthogonal for continuous QPTs makes it difficult (if not impossible) to extract physical information solely from ground states themselves. Conventionally, condensed matter physicists and field theorists focus on spectra and correlation functions. Therefore, it is somewhat surprising to see that simply partitioning a system into two parts and quantifying entanglement between them reveal highly nontrivial information about QPTs. The intrinsic irreversibility due to information loss along RG flows may also be revealed solely from ground states [4, 15–17]. In the fidelity approach [10], it is necessary to put the whole system on a finite chain, and observe how the fidelity scales with system sizes as the thermodynamic limit is approached, in order to extract physical information. The difference between entanglement measures and the fidelity approach lies in the fact that for the former different entanglement measures need to be devised to detect QPTs [7], whereas the latter succeeds in detecting QPTs for quantum spin chains, regardless of what order is present. The philosophy behind this is that bipartite entanglement measures involve partitions and some information is lost due to the fact that the whole is not simply the sum of the parts, whereas in the fidelity approach, a system is treated as a whole from the starting point.

In this communication, we analyze the fidelity per lattice site for the one-dimensional quantum Ising model in a transverse field near the critical point. It is found that, in the thermodynamic limit, the fidelity per lattice site is singular, and the derivative of its logarithmic function with respect to the transverse field strength (the control parameter) is logarithmically divergent at the critical point. A finite-size scaling analysis is carried out for systems of different sizes, and the scaling behavior is confirmed numerically, consistent with the conformal invariance at the critical point. This allows us to extract the correlation length critical exponent. We have also performed numerics to confirm the universality, i.e., the correlation length critical exponent does not depend on either the anisotropic parameter or the transverse field strength.

The fidelity per lattice site for the quantum XY spin chain

The quantum XY spin chain is described by the Hamiltonian

$$H = - \sum_{j=-M}^M \left(\frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y + \lambda \sigma_j^z \right). \quad (1)$$

Here σ_j^x , σ_j^y and σ_j^z are the Pauli matrices at the j th lattice site. The parameter γ denotes an anisotropy in the nearest-neighbor spin–spin interaction, whereas λ is an external magnetic field. The Hamiltonian (1) may be exactly diagonalized [18, 19] as $H = \sum_k \Lambda_k (c_k^\dagger c_k - 1)$, where $\Lambda_k = \sqrt{(\lambda - \cos(2\pi k/L))^2 + \gamma^2 \sin^2(2\pi k/L)}$, with c_k and c_k^\dagger denoting free fermionic modes and $L = 2M + 1$. The ground state $|\psi\rangle$ is the vacuum of all fermionic modes defined by $c_k |\psi\rangle = 0$, and may be written as $|\psi\rangle = \prod_{k=1}^M (\cos(\theta_k/2) |0\rangle_k |0\rangle_{-k} - i \sin(\theta_k/2) |1\rangle_k |1\rangle_{-k})$, where $|0\rangle_k$ and $|1\rangle_k$ are, respectively, the vacuum and single excitations of the k th mode, and θ_k is defined by $\cos \theta_k = (\cos(2\pi k/L) - \lambda)/\Lambda_k$. Therefore, the fidelity F for two different ground states $|\psi(\lambda, \gamma)\rangle$ and $|\psi(\lambda', \gamma)\rangle$ takes the form

$$F(\lambda, \lambda'; \gamma) = \prod_{k=1}^M \cos \frac{\theta_k - \theta'_k}{2}, \quad (2)$$

where the prime denotes that the corresponding variables take their values at λ' . Obviously, $F = 1$ if $\lambda = \lambda'$. Generically, $\cos \frac{\theta_k - \theta'_k}{2} < 1$; therefore, the fidelity (2) decays very fast when λ separates from λ' .

Now let us introduce a fundamental scaling parameter, i.e. the fidelity per lattice site, $d(\lambda, \lambda'; \gamma)$. For a large but finite L , the fidelity scales as d^L , with some scaling parameter d depending on λ and λ' , due to the symmetry under translation. Formally, in the thermodynamic limit, $d(\lambda, \lambda')$ may be defined as

$$\ln d(\lambda, \lambda'; \gamma) = \lim_{L \rightarrow \infty} \ln F(\lambda, \lambda'; \gamma)/L. \quad (3)$$

The fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ enjoys some properties inherited from the fidelity: (1) symmetry under interchange $\lambda \longleftrightarrow \lambda'$, (2) $d(\lambda, \lambda; \gamma) = 1$ and (3) $0 \leq d(\lambda, \lambda'; \gamma) \leq 1$.

In the thermodynamic limit, the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ for the quantum XY model takes the form

$$\ln d(\lambda, \lambda'; \gamma) = \frac{1}{2\pi} \int_0^\pi d\alpha \ln \mathcal{F}(\lambda, \lambda'; \gamma; \alpha), \quad (4)$$

where

$$\mathcal{F}(\lambda, \lambda'; \gamma; \alpha) = \cos[\vartheta(\lambda; \gamma; \alpha) - \vartheta(\lambda'; \gamma; \alpha)]/2, \quad (5)$$

with

$$\cos \vartheta(\lambda; \gamma; \alpha) = (\cos \alpha - \lambda)/\sqrt{(\cos \alpha - \lambda)^2 + \gamma^2 \sin^2 \alpha}. \quad (6)$$

A notable feature of the fidelity per lattice site (4) is that, besides $d(\lambda, \lambda'; \gamma) = d(\lambda', \lambda; \gamma)$ and $d(\lambda, \lambda; \gamma) = 1$, it even detects the duality between two phases $\lambda > 1$ and $\lambda < 1$ for the quantum Ising model in a transverse field ($\gamma = 1$) [19], since it satisfies $d(\lambda, \lambda'; 1) = d(1/\lambda, 1/\lambda'; 1)$.

It has been shown [10] that the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ exhibits a pinch point at (1, 1), i.e. an intersection of two singular lines $\lambda = 1$ and $\lambda' = 1$, for the quantum Ising model in a transverse field ($\gamma = 1$). In figure 1, we plot the scaling parameter $d(\lambda, \lambda'; \gamma)$ against λ for different values of λ' and γ . One observes the continuity, as it should be for continuous QPTs. Let us focus on the quantum Ising universality class with the critical line $\gamma \neq 0$ and $\lambda_c = 1$. There is only one (second-order) critical point $\lambda_c = 1$ separating two gapful phases: (spin reversal) Z_2 symmetry breaking and symmetric phases. The order parameter, i.e., magnetization $\langle \sigma^x \rangle$ is non-zero for $\lambda < 1$, and otherwise zero. At the critical point, the correlation length $\xi \sim |\lambda - \lambda_c|^\nu$ with $\nu = 1$ [19]. Our purpose is to extract the correlation length critical exponent by performing a finite-size scaling analysis for $d(\lambda, \lambda'; \gamma)$.

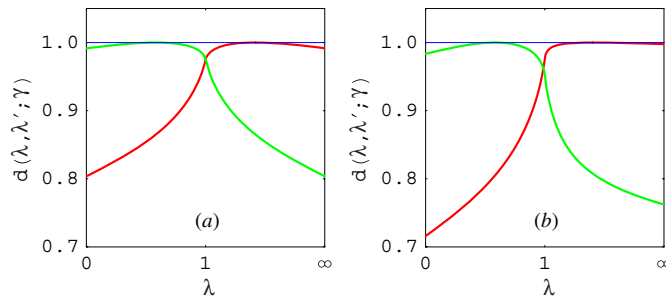


Figure 1. The fidelity per lattice site $d(\lambda, \lambda'; \gamma)$, extracted from the fidelity for two ground states $|\psi(\lambda)\rangle$ and $|\psi(\lambda')\rangle$ of the quantum Ising model in a transverse field, is regarded as a function of λ for some fixed values of λ' and γ . It is continuous but not analytic at $\lambda_c = 1$. (a) The (thick) red line is for $\lambda' = 2, \gamma = 1$, which touches the blue (dashed) line at $\lambda = 2$ and the green (thin) line is for $\lambda' = 1/2, \gamma = 1$, touching the blue (dashed) line at $\lambda = 1/2$. The mirror symmetry between two curves results from the duality. (b) The green (thin) line is for $\lambda' = 1/2, \gamma = 1/2$, touching the blue (dashed) line at $\lambda = 1/2$ and the red (thick) line is for $\lambda' = 2, \gamma = 1/2$, touching the blue (dashed) line at $\lambda = 2$. No mirror symmetry for $\gamma \neq 1$.

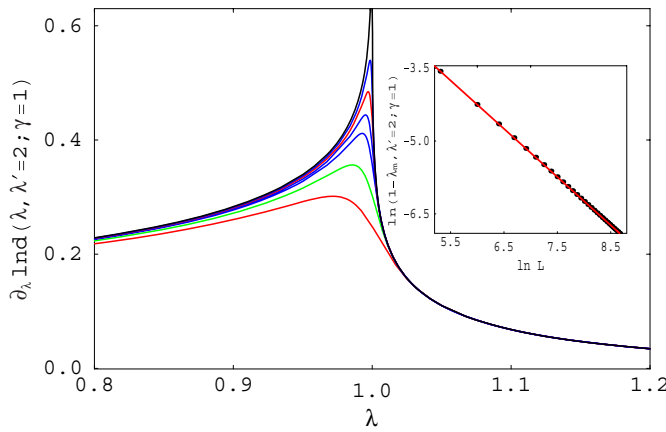


Figure 2. Main: the logarithmic divergence near the critical point $\lambda_c = 1$ is analyzed. This is achieved by considering $\partial_\lambda \ln d(\lambda, \lambda' = 2; \gamma = 1)$ as a function of the transverse field strength λ . The curves shown correspond to different lattice sizes $L = 201, 401, 1201, 2001, 4001, \infty$. The maximum gets more pronounced with the system size increasing. Inset: the position of maximum approaches the critical point $\lambda_c = 1$ as $\lambda_m \sim 1 - 5.52233L^{-0.99321}$.

Finite-size scaling

In order to quantify the drastic change of the ground-state wavefunctions when the system undergoes a QPT at the critical point $\lambda_c = 1$, we evaluate the derivative of $\ln d(\lambda, \lambda'; \gamma)$ with respect to λ . In the thermodynamic limit, $\ln d(\lambda, \lambda'; \gamma)$ is logarithmically divergent at the critical point $\lambda_c = 1$:

$$\frac{\partial \ln d(\lambda, \lambda'; \gamma)}{\partial \lambda} = k_1 \ln|\lambda - \lambda_c| + \text{constant}, \quad (7)$$

where the prefactor k_1 is non-universal in the sense that it depends on λ' and γ . The numerical results are plotted in figure 2 for $\lambda' = 2$ and $\gamma = 1$. The least square method yields $k_1 \approx -0.079742$. For systems of finite sizes L 's, there are no divergence in the derivatives of

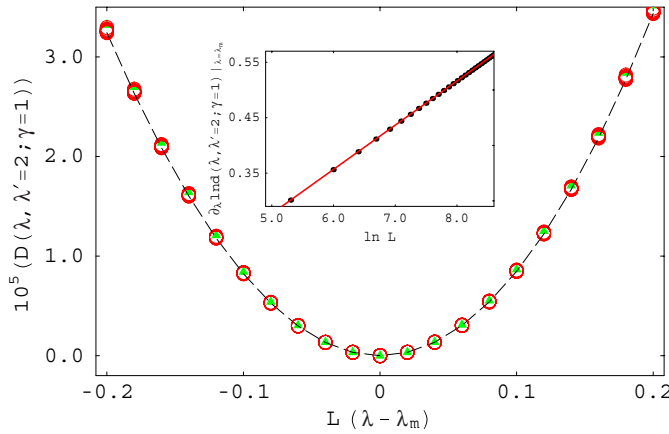


Figure 3. Main: a finite-size scaling analysis is carried out for a quantity defined as $D(\lambda, \lambda'; \gamma) = 1 - \exp[\partial_\lambda \ln d(\lambda, \lambda'; \gamma) - \partial_\lambda \ln d(\lambda, \lambda'; \gamma)|_{\lambda=\lambda_m}]$. According to the finite-size scaling ansatz in the case of logarithmic divergences, one expects that $D(\lambda, \lambda'; \gamma)$ is a function of $L(\lambda - \lambda_m)$. Indeed, all the data from $L = 801$ up to $L = 4001$ collapse on a single curve. This shows that the system at the critical point is scaling invariant (and thus conformally invariant) and that the correlation length critical exponent ν is 1. Inset: the peak value of $\partial_\lambda \ln d(\lambda, \lambda' = 2; \gamma = 1)$ at λ_m diverges as the system size increases, leading to $k_2 \approx 0.079\,7726$.

$\ln d(\lambda, \lambda'; \gamma)$ with respect to λ , since the second-order QPT only occurs in the thermodynamic limit. Instead, some pronounced peaks occur at the so-called quasi-critical points λ_m that approach the critical value as $\lambda_m \sim 1 - 5.522\,33L^{-0.993\,21}$, with the peak values logarithmically diverging with the increasing system size L :

$$\left. \frac{\partial \ln d(\lambda, \lambda'; \gamma)}{\partial \lambda} \right|_{\lambda=\lambda_m} = k_2 \ln L + \text{constant}, \tag{8}$$

where the non-universal prefactor k_2 takes the value $k_2 \approx 0.079\,773$. The scaling ansatz in the systems exhibiting logarithmic divergences [20] requires that the absolute value of the ratio k_1/k_2 is the correlation length critical exponent ν . In this case, $|k_1/k_2| \sim 0.999\,613$, very close to the exact value 1.

In the case of logarithmic divergences, a proper scaling ansatz has been addressed in [20]. Taking into account the distance of the maximum of $\partial_\lambda \ln d(\lambda, \lambda'; \gamma)$ from the critical point, we choose to plot $1 - \exp[\partial_\lambda \ln d(\lambda, \lambda'; \gamma) - \partial_\lambda \ln d(\lambda, \lambda'; \gamma)|_{\lambda=\lambda_m}]$ as a function of $L(\lambda - \lambda_m)$ for different system sizes L 's. All the data for different L 's collapse onto a single curve. The numerical results for the size ranging from $L = 201$ up to $L = 4001$ are plotted in figure 3. All these indicates that the system is scaling invariant, i.e. $\xi/L = \xi'/L'$ (and thus conformally invariant), and that the correlation length critical exponent $\nu = 1$.

Universality

As is well known, the quantum XY chain belongs to the same quantum Ising universality class for non-zero γ , with the same critical exponents. To confirm the universality, we need to check the scaling behaviors for different values of γ . For $\lambda' = 2$ and $\gamma = 1/2$, in the thermodynamic limit, it takes the form (7) with $k_1 \approx -0.157\,162$, as long as the control parameter is close to the critical point, whereas for a system of finite size, it takes the form (8) with $k_2 \approx 0.157\,176$. Thus, the absolute value of the ratio k_1/k_2 is $|k_1/k_2| = 0.999\,910$.

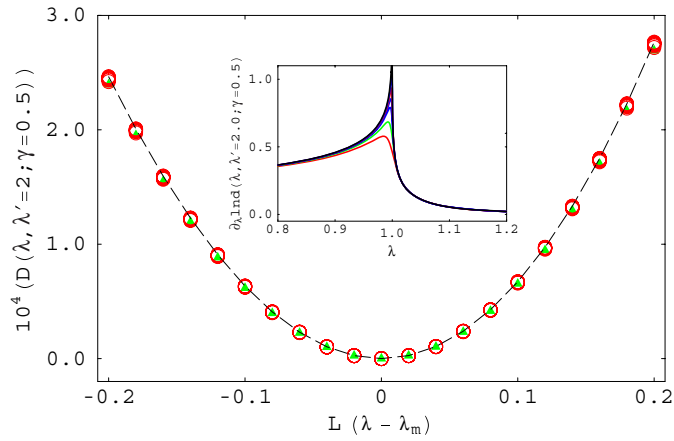


Figure 4. The universality hypothesis for the fidelity per lattice site, extracted from the fidelity for two ground states of the quantum Ising model in a transverse field, is checked against different values of γ and λ' . Main: in this case, we have chosen $\gamma = 1/2$ and L ranging from 2801 up to 6001. All the data collapse, consistent with the fact that the correlation length critical exponent ν is 1. The inset shows that the derivative of the logarithmic function of the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ with respect to λ for $\lambda' = 2$ and $\gamma = 1/2$ is logarithmically divergent at $\lambda_c = 1$, with $\lambda_m \sim 1 - 3.23906L^{-1.01135}$.

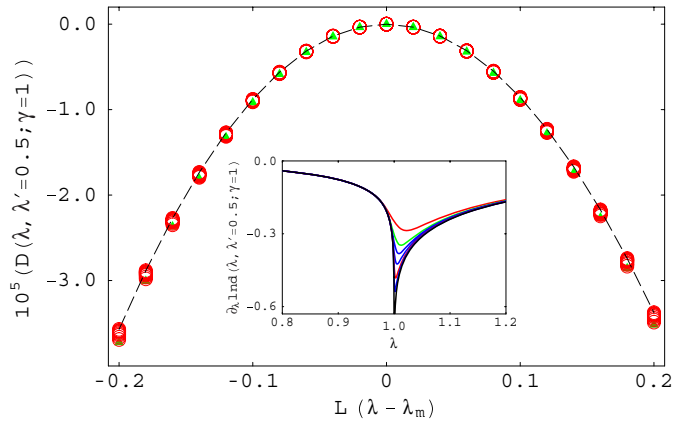


Figure 5. The universality for the fidelity per lattice site is checked against different values of γ and λ' . Main: in this case we have chosen $\gamma = 1$ and L ranging from 2801 up to 6001. Consistent with the universality hypothesis for the quantum Ising model in a transverse field, all the data collapse, indicating that the correlation length critical exponent ν is 1. The inset shows that the derivative of the logarithmic function of the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ with respect to λ for $\lambda' = 1/2$ and $\gamma = 1$ is logarithmically divergent at $\lambda_c = 1$, with $\lambda_m \sim 1 + 3.50186L^{-0.94107}$.

Figure 4 shows that all the data for different L 's collapse onto a single curve. We also plot the derivative of the logarithmic function of the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ with respect to λ for $\lambda' = 2$ and $\gamma = 1/2$ (see the inset in figure 4). All the above results show that the critical exponent $\nu = 1$.

Besides γ , the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ also depends on the control parameter λ' . For $\lambda' = 1/2$ and $\gamma = 1$, in the thermodynamic limit, the derivative of the logarithmic function of the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ with respect to λ still takes the form (7), with

$k_1 \approx 0.083\,005$, as long as the control parameter is close to the critical point, whereas for a system of finite size, it takes the form (8), with $k_2 \approx -0.083\,007$. Thus, the absolute value of the ratio k_2/k_1 is $|k_2/k_1| = 0.999\,975$, again close to the exact value 1. Similarly, all the data for different L 's collapse onto a single curve, as shown in figure 5. In the inset, we plot the derivative of the logarithmic function of the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$ with respect to λ for $\lambda' = 1/2$ and $\gamma = 1$. Therefore, we have demonstrated that the universality hypothesis is valid for the fidelity per lattice site $d(\lambda, \lambda'; \gamma)$.

Discussions and conclusions

As a basic notion of quantum information science, fidelity may be used to detect QPTs in condensed matter systems. Remarkably, an intimate connection exists between RG flows, QPTs and the fidelity per lattice site [10]. The latter is well defined in the thermodynamic limit, in sharp contrast to the fidelity itself that always vanishes for continuous QPTs. Different from a bipartite entanglement measure, the fidelity approach does not involve the partition of the whole system into different parts, and the system is treated as a *whole* from the starting point. In some sense, such a difference may be counted as the contribution from multipartite entanglement. Therefore, one may expect that the fidelity approach possesses significant advantage over the conventional bipartite entanglement approach [21].

Another feature worth to be mentioned is that the fidelity per lattice site is simple to be evaluated in the matrix product state (MPS) representation [10]. On the other hand, many efficient numerical algorithms are now available due to the latest developments in classical simulation of quantum systems [22–24]. This makes it practical to determine all information including stable and unstable fixed points along RG flows [10], and to extract critical exponents from the fidelity per lattice site, as shown for the quantum Ising model in a transverse field. In this regard, algorithms for periodic boundary conditions [22] and infinite systems [23] are powerful enough to extract meaningful information for critical systems.

In summary, we have performed a finite size scaling analysis for the fidelity per lattice site, whose analytical expression has been extracted from the fidelity for two ground states corresponding to different values of the control parameter for the one-dimensional quantum Ising model in a transverse field near the critical point. In the thermodynamic limit, the logarithmic divergence of the derivative of the fidelity per lattice site with respect to the transverse field strength is demonstrated numerically, consistent with the conformal invariance at the critical point. This makes it possible to extract the correlation length critical exponent. The latter turns out to be universal in the sense that the correlation length critical exponent thus extracted does not depend on either the anisotropic parameter γ or the transverse field strength λ .

Acknowledgments

We thank Sam Young Cho, John Fjaerestad and Jon Links for helpful discussions and comments.

References

- [1] Sachdev S 1999 *Quantum Phase Transitions* (Cambridge: Cambridge University Press)
- [2] Preskill J 2000 *J. Mod. Opt.* **47** 127
- [3] Osborne T J and Nielsen M A 2002 *Phys. Rev. A* **66** 032110
Osterloh A, Amico L, Falci G and Fazio R 2002 *Nature* **416** 608

- [4] Vidal G, Latorre J I, Rico E and Kitaev A 2003 *Phys. Rev. Lett.* **90** 227902
Korepin V E 2004 *Phys. Rev. Lett.* **92** 096402
Levine G C 2004 *Phys. Rev. Lett.* **93** 266402
Rafael G and Moore J E 2004 *Phys. Rev. Lett.* **93** 260602
Calabrese P and Cardy J 2004 *J. Stat. Mech.* P06002
- [5] Verstraete F, Martin-Delgado M A and Cirac J I 2004 *Phys. Rev. Lett.* **92** 087201
Dür W, Hartmann L, Hein M, Lewenstein M and Briegel H J 2005 *Phys. Rev. Lett.* **94** 097203
- [6] Kitaev A and Preskill J 2006 *Phys. Rev. Lett.* **96** 110404
Levin M and Wen X-G 2006 *Phys. Rev. Lett.* **96** 110405
- [7] Barnum H, Knill E, Ortiz G, Somma R and Viola L 2004 *Phys. Rev. Lett.* **92** 107902
- [8] Amico L, Fazio R, Osterloh A and Vedral J V 2007 arXiv:quant-ph/0703044
- [9] Zanardi P and Paunković N 2006 *Phys. Rev. E* **74** 031123
- [10] Zhou H-Q and Barjaktarević J P 2008 Fidelity and quantum phase transitions *J. Phys. A: Math. Theor.* **41** 412001 (cond-mat/0701608)
- [11] Zanardi P, Cozzini M and Giorda P 2006 (arXiv:cond-mat/0606130)
Oelkers N and Links J 2007 *Phys. Rev. B* **75** 115119 arXiv:cond-mat/0611510
Cozzini M, Ionicioiu R and Zanardi P 2006 arXiv:cond-mat/0611727
Buonsante P and Vezzani A 2006 arXiv:cond-mat/0612590
You W-L, Li Y-W and Gu S-J 2007 arXiv:quant-ph/0701077
- [12] Wolf M, Ortiz G, Verstraete F and Cirac J I 2006 *Phys. Rev. Lett.* **97** 110403
- [13] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [14] Furukawa S, Misguich G and Oshikawa M 2006 *Phys. Rev. Lett.* **96** 047211
- [15] Latorre J I, Lütken C A, Rico E and Vidal G 2005 *Phys. Rev. A* **71** 034301
- [16] Orús R 2005 *Phys. Rev. A* **71** 052327
- [17] Zhou H-Q, Barthel T, Fjærestad J O and Schollwöck U 2006 *Phys. Rev. A* **74** 050305
- [18] Lieb E, Schultz T and Mattis D 1961 *Ann. Phys.* **60** 407
- [19] Pfeuty P 1970 *Ann. Phys.* **57** 79
- [20] Barber M N 1983 *Phase Transition and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (London: Academic) p 145
- [21] Zhou H-Q 2007 Renormalization group flows and quantum phase transitions: fidelity versus entanglement arXiv:0704.2945
- [22] Verstraete F, Porras D and Cirac J I 2004 *Phys. Rev. Lett.* **93** 227205
- [23] Vidal G 2006 arXiv:cond-mat/0605597
Singh S, Zhou H-Q and Vidal G 2007 arXiv:cond-mat/0701427
- [24] McCulloch I P 2007 arXiv:cond-mat/0701428